Misinformation Control in the Internet of Battlefield Things: A Multiclass Mean-field Game

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PROOF OF CONVERGENCE

In this section, we extend the convergence results of the finite state mean-field games in [1] to the case of multiclass agents and when the transitional rate is a function of the control as well as the mean field. We show the conditions under which the cost and distribution functions of the $N+1$ player game converges to corresponding functions in the mean-field game.

At time 0, it is assumed that the players of class (i,k) are distributed according to a given initial distribution $\boldsymbol{\nu}_{ik} = (\nu_{ik}^{l})_{l \in \mathcal{S}}$. Thus, the number of players in each state are distributed according to a multinomial distribution with parameter $\boldsymbol{\nu}_{ik}$.

In order to prove convergence, we rely on the following properties of our game

Proposition 1. In our problem,

1) The transitional rate $G_{ik}^{jl}(\alpha_{ik}(t), \Theta(t))$ is a Lipchitz function of $\alpha_{ik}(t)$ for all i, k.
2) The best response $\alpha_{ik}^{*}(\Delta_l u, \boldsymbol{\gamma}(t))$ is Lipschitz in $\Delta_l u$, $\Theta(t)$, $\eta(t)$, and $m_{ik}(t)$ $\forall$ $(i,k) \in \mathcal{C}$ provided that the cost $v_{ik}(l, \alpha_{ik}^{l}(t), \boldsymbol{\gamma}(t))$ is strongly convex.
3) The transitional rate $G_{ik}^{jl}(\alpha_{ik}^{*}, \Theta(t))$ is Lipchitz in $\Delta_j u$ and $\Theta(t)$.
4) The function $h(\Delta_l u, \boldsymbol{\gamma}(t), l)$ is Lipchitz in $\Delta_l u$, $\Theta(t)$, and $\eta(t)$.

Proof. 1) The transitional rate $G_{ik}^{jl}(\alpha_{ik}^{l}(t), \Theta(t))$) is only a function of $\alpha_{ik}^{j}(t)$ when $j = \{S_N, S_T\}$, and, in this case, it is a linear function of $\alpha_{ik}^{j}(t)$ and therefore Lipschitz in $\alpha_{ik}^{j}(t)$.

2) Proving that the best response $\alpha_{ik}^{*}(\Delta_l u, \boldsymbol{\gamma}(t))$ is Lipschitz in $\Delta_l u$, $\Theta(t)$ $\eta(t)$ can be shown using a similar proof as [1, Proposition 1] of and using the fact the transitional rate $G_{ik}^{jl}(\alpha_{ik}(t), \Theta(t))$ is Lipchitz in $\alpha_{ik}(t)$. A consequence is that the best response $\alpha_{ik}^{*}(\Delta_l u, \boldsymbol{\gamma}(t))$ is Lipchitz in $m_{ik}(t)$ $\forall (i,k) \in \mathcal{C}$ since both $\Theta(t)$ and $\eta(t)$ are linear functions of $m_{ik}(t)$ $\forall (i,k) \in \mathcal{C}$. However, the proof relies on the assumption that that cost $v_{ik}(l, \alpha_{ik}^{l}(t), \boldsymbol{\gamma}(t))$ is strongly convex w.r.t $\alpha_{ik}^{l}(t)$. In our problem, the second derivative of the cost w.r.t. $\alpha_{ik}^{l}(t)$ is $Q_{ik}(\boldsymbol{\gamma}(t))^2$ which can be zero if $2\Theta(t) + \rho(t) = \frac{k+1-2\lambda_{ik}(t)}{k}$. Thus, in order to ensure that the cost is strongly convex, the values of $Q_{ik}(\boldsymbol{\gamma}(t))$ are scaled such that the resulting values are always positive. Let $Q'_{ik}(\boldsymbol{\gamma}(t))$ the scaled valued. $Q'_{ik}(\boldsymbol{\gamma}(t))$ can be possibly defined as $Q'_{ik}(\boldsymbol{\gamma}(t)) = Q_{ik}(\boldsymbol{\gamma}(t)) + S_k$ where S_k is the scaling factor and is given by $S_k = 3k$.

3) In order to prove this property, we first note that the transitional rate is only a function of $\alpha_{ik}(t)$ and $\Theta(t)$ only when the state is in $\{S_T, S_N\}$. We consider the transitional rate $G_{ik}^{S_T I}(\alpha_{ik}^{S_T}(t), \Theta(t))) = \alpha_{ik}^{S_T}(t) R_{ik}(\Theta(t))$ and compute its partial derivative with respect to $\Theta(t)$

$$\frac{\partial G_{ik}^{S_T I}}{\partial \Theta(t)}(\alpha_{ik}^{S_T *}(t)) = \frac{\partial}{\partial \Theta(t)} \alpha_{ik}^{S_T *}(t)\Theta(t) = \Theta(t) \frac{\partial}{\partial \Theta(t)} \alpha_{ik}^{S_T *}(t) + \alpha_{ik}^{S_T *}(t). \tag{1}$$

The partial derivative $\frac{\partial}{\partial \Theta(t)} \alpha_{ik}^{S_T *}(t)$ is bounded since $\alpha_{ik}^{S_T *}(t)$ is Lipschitz in $\Theta(t)$ according to 2). Further, $\Theta(t)$ and $\alpha_{ik}^{S_T *}(t)$ are bounded by 1. Thus, $\frac{\partial G_{ik}^{S_T I}}{\partial \Theta(t)}(\alpha_{ik}^{S_T *}(t))$ and the transitional rate $G_{ik}^{S_T I}$ is Lipschitz in $\Theta(t)$.

Further, $G_{ik}^{S_T I}(\alpha_{ik}^{S_T *}(t), \Theta(t))$ is a linear function of $\alpha_{ik}^{S_T *}(t)$ and therefore is Lipschitz in $\Delta_l u$ for all l since $\alpha_{ik}^{S_T *}(t)$ is Lipschitz in $\Delta_l u$.

This property can be proved for the remaining transitional probabilities using a similar method.

4) This property easily follows from 1) and 3).

□

Next, we use the following useful property from [1, Proposition 7] which holds for the solution $u_{ik}^{N,\boldsymbol{n},l}$ to our HJ equations.

Remark 1. Let $u_{ik}^{N,\boldsymbol{n},l}(t)$ be the solution of the HJ equations of the finite IoBT game. Then, there exists $C > 0$ and $T^* > 0$ such that for $0 < T < T^*$,

$$\max_{r,v} ||u_{ik}^{N,\boldsymbol{n}+\boldsymbol{e}_{ik}^{rv},l}(t) - u_{ik}^{N,\boldsymbol{n},l}(t)|| \leq \frac{2C}{N}, \tag{2}$$

where the norm ||.|| used is the ∞ norm.

The property can be proved for our problem using a similar proof of [1, Proposition 7] and using the property 2) from Proposition 1.

Further in this part, we replace $h(\Delta_l \boldsymbol{u}_{ik}, \boldsymbol{\gamma}(t), l)$ by $h(\Delta_l \boldsymbol{u}_{ik}, \boldsymbol{m}(t), l)$ and $h(\Delta_l u_{ik}^{N,\boldsymbol{n}}, \boldsymbol{\gamma}_N(t), l)$ by $h(\Delta_l u_{ik}^{N,\boldsymbol{n}}, \boldsymbol{m}^N(s), l)$ since both $\Theta(t)$ and $\eta(t)$ are linear functions of $\boldsymbol{m}(t)$. Similarly, both $\Theta_N(t)$ and $\eta_N(t)$ are both functions of $(\boldsymbol{n}_{ik}(t))_{(i,k)\in\mathcal{C}}$ where $\boldsymbol{m}(t) = (\boldsymbol{m}^{ik}(t))_{(i,k)\in\mathcal{C}}$ and $\boldsymbol{m}^N(s) = (\frac{\boldsymbol{n}_{ik}(s)}{N_{ik}})_{(i,k)\in\mathcal{C}}$. We present the convergence results in the following theorem.

Theorem 1. *Let T^* be as in Remark 1. There exists a constant $\bar{C}$ independent of N, for which, if $T < T^*$, satisfies $\mu = T\bar{C} < 1$ then*

$$\sum_{ik} V_{ik}^N(t) + W_{ik}^N(t) \leq \frac{\bar{C}}{1-\mu}\frac{1}{N_{\max}}, \tag{3}$$

for all $t \in [0, T]$, where

$N_{\max} = \max_{(r,v)\in\mathcal{C}} N_{rv}$, $W_{ik}^N(t) = \mathbb{E}\Big[||\boldsymbol{u}_{ik}(t) - \boldsymbol{u}_{ik}^{N,\boldsymbol{n}}(t)||^2\Big]$, $V_{ik}^N(t) = \mathbb{E}(||\frac{\boldsymbol{n}_{ik}(t)}{N_{ik}} - \boldsymbol{m}_{ik}(t)||)^2$, *$\boldsymbol{m}_{ik}(t)$ and $\boldsymbol{u}_{ik}(t)$ are the meanfield and cost functions at the MFE, $\boldsymbol{n}_{ik}(t)$ and $\boldsymbol{u}_{ik}^{N,\boldsymbol{n}}(t)$ are the equilibrium distribution and cost value of $N+1$ player game .*

Proof. The proof of Theorem 1 relies on the following two lemmas.

Lemma 2. *Define T^* defined as done in Remark 1, then, there exists C_1 such that*

$$W_{ik}^N(t) \leq \frac{C_1}{N} + C_1\mathbb{E}\int_t^T \Big(W_{ik}^N(s) + \sum_{(r,v)\in\mathcal{C}} V_{rv}^N(s)\Big)ds. \tag{4}$$

Proof. See appendix ☐

Lemma 3. *Define T^* as done in Remark 1, then, there exists C_2 such that*

$$V_{ik}^N(t) \leq C_2\mathbb{E}\int_0^t (V_{ik}^N(s) + W_{ik}^N(s) + V_{yz}^N(s))ds + \frac{C_2}{N_{\max}}, \tag{5}$$

where $(y, z) = \arg\max_{(r,v)} V_{rv}(t)$ and $N_{\max} = \max_{(r,v)\in\mathcal{C}} N_{rv}$.

Proof. See appendix. ☐

By adding (4) and (5) for all (i,k), we have

$$\begin{aligned}\sum_{ik} W_{ik}^N(t) + \sum_{ik} V_{ik}^N(t) &\leq C_1 \mathbb{E}\int_0^t \sum_{ik}\Big(W_{ik}^N(s) + \sum_{rv} V_{rv}^N(s)\Big) + \frac{C_1|\mathcal{C}|}{N} ds \\ &\quad + C_2\mathbb{E}\int_t^T \sum_{ik}\Big(W_{ik}^N(s) + V_{ik}^N(s) + V_{yz}^N(s)\Big) + \frac{C_2|\mathcal{C}|}{N_{\max}}, \\ &\leq \bar{C}\mathbb{E}\int_0^T \sum_{ik} V_{ik}^N(s) + W_{ik}^N(s) + \frac{\bar{C}}{N_{\max}}, \qquad (6)\end{aligned}$$

where $\bar{C} = \max\{C_1|\mathcal{C}|, C_2+1, C_2|\mathcal{C}|\}$.
Let $W_{ik}^N + V_{ik}^N = \max_{0\leq t\leq T} W_{ik}^N(t) + V_{ik}^N(t)$. Then,

$$\sum_{ik} W_{ik}^N(t) + V_{ik}^N(t) \leq \sum_{ik} W_{ik}^N + V_{ik}^N \leq \bar{C}T\sum_{ik} W_{ik}^N + V_{ik}^N + \frac{\bar{C}}{N_{\max}} \leq \frac{\bar{C}}{(1-\mu)N_{\max}}, \qquad (7)$$

where $\mu = \bar{C}T$. Thus, the value function and the proportion of nodes converges uniformly in distribution to the meanfield case. Thus, the meanfield equilibrium constitutes an ϵ Nash equilibrium as demonstrated in [2].

Appendix

Appendix A: Proof of Lemma 2

Let $W_{ik}^N(l,t) = \mathbb{E}\Big[(\boldsymbol{u}_{ik}^l(t) - \boldsymbol{u}_{ik}^{N,\boldsymbol{n},l}(t))^2\Big]$. Thus, $W_{ik}^N(t) = \max_{l\in\mathcal{S}} W_{ik}^N(l,t)$. To prove the lemma, we apply Dynkin formula on functions of the process $(l,\boldsymbol{n}_{ik})$. First, we define the infinitesimal generator acting on a function of the process $(l,\boldsymbol{n}_{ik})$ $\varphi : (\mathcal{S}, \mathcal{N}^{\mathcal{S}}, [0,T]) \to \mathbb{R}$ as

$$\begin{aligned}A_{ik}\varphi(l,\boldsymbol{n}_{ik},s) &= \sum_{j\in\mathcal{S}} G_{lj}^{ik}(\alpha_{ik}^{N,l}(s))[\varphi(j,\boldsymbol{n}_{ik}(s),s) - \varphi(l,\boldsymbol{n}_{ik}(s),s)] \\ &\quad + \sum_{j\in\mathcal{S}}\sum_{y\in\mathcal{S}} n_{ik}^y G_{yj}^{N,ik}(\alpha_{ik}^{N,y}(s))[\varphi(l,\boldsymbol{n}_{ik}(s)+\boldsymbol{e}_{jy},s) - \varphi(l,\boldsymbol{n}_{ik}(s),s)], \qquad (8)\end{aligned}$$

where $\alpha_{ik}^{N,y} = \alpha_{ik}^{N,y*}(\boldsymbol{\gamma}_N(\boldsymbol{n}(t)+\boldsymbol{e}_{ly}^{ik}), \Delta_y u_{ik}^{N,\boldsymbol{n}(t)+\boldsymbol{e}_{ly}^{ik}})$ for $(i,k)\neq(i',k')$ ((i',k') is the class of the reference player) and $\alpha_{ik}^{N,y} = \alpha_{ik}^{N,y*}(\boldsymbol{\gamma}_N(\boldsymbol{n}(t)-\boldsymbol{e}_{y}^{ik}), \Delta_y u_{ik}^{N,\boldsymbol{n}(t)-\boldsymbol{e}_{y}^{ik}})$ for $(i,k)\neq(i',k')$ are the equilibrium acceptance probabilities for the finite IoBT game. Using Dynkin formula, we have

$$\mathbb{E}[\varphi(l_{ik}(T),\boldsymbol{n}_{ik}(T),T) - \varphi(l_{ik}(t),\boldsymbol{n}_{ik}(t),t)] = \mathbb{E}\Big[\int_t^T \frac{d\varphi}{dt}(l_{ik}(s),\boldsymbol{n}_{ik}(s),s) + A_{ik}\varphi(l_{ik}(s),\boldsymbol{n}_{ik}(s),s)ds\Big], \quad (9)$$

where $l_{ik}(s)$ is the state of the reference player at time s.

Next, we define $\varphi_l(j,\boldsymbol{n}_{ik}(t),t)=(\boldsymbol{u}^l_{ik}(t)-\boldsymbol{u}^{N,\boldsymbol{n},l}_{ik}(t))^2$. Using (9), we have

$$\begin{aligned}
&W^N_{ik}(l,t)-W^N_{ik}(l,T)=-\mathbb{E}\Big[(\boldsymbol{u}^{N,\boldsymbol{n},l}_{ik}(t)-\boldsymbol{u}^l_{ik}(t))^2\Big]+\mathbb{E}\Big[(\boldsymbol{u}^{N,\boldsymbol{n},l}_{ik}(T)-\boldsymbol{u}^l_{ik}(T))^2\Big]\\
&=\mathbb{E}\int_t^T 2(\boldsymbol{u}^{N,\boldsymbol{n},l}_{ik}(s)-\boldsymbol{u}^l_{ik}(s))\frac{d}{ds}(\boldsymbol{u}^{N,\boldsymbol{n},l}_{ik}(s)-\boldsymbol{u}^l_{ik}(s))ds+\int_t^T\sum_{jy}n^y_{ik}G^{N,ik}_{yj}(\alpha^{N,y}_{ik}(s))[\varphi(l,\boldsymbol{n}_{ik}(s)+\boldsymbol{e}^{ik}_{jy},s)-\varphi(l,\boldsymbol{n}_{ik}(s),s)]\\
&=\mathbb{E}\int_t^T 2(\boldsymbol{u}^{N,\boldsymbol{n},l}_{ik}(s)-\boldsymbol{u}^l_{ik}(s))\Big(\sum_{y,j}\eta^l_{ik}(y,j,\boldsymbol{n})(u^{N,\boldsymbol{n}+\boldsymbol{e}^{ik}_{jy},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))-h(\Delta_l u^{N,\boldsymbol{n}}_{ik},\boldsymbol{m}^N(s),l)+h(\Delta_l\boldsymbol{u}_{ik},\boldsymbol{m}(s),l)ds,\\
&+\mathbb{E}\int_t^T\sum_{jy}n^y_{ik}G^{N,ik}_{yj}(\alpha^{N,y}_{ik}(s))(u^{N,\boldsymbol{n}+\boldsymbol{e}^{ik}_{jy},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))^2-(u^{N,\boldsymbol{n},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))^2),\\
&=\mathbb{E}\int_t^T\sum_{jy}n^y_{ik}G^{N,ik}_{yj}(\alpha^{N,y}_{ik}(s))(u^{N,\boldsymbol{n}+\boldsymbol{e}^{ik}_{jy},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))^2ds+\mathbb{E}\int_t^T(2(u^{N,\boldsymbol{n},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))(h(\Delta_l\boldsymbol{u}_{ik},\boldsymbol{m}(s),l),\\
&\quad -h(\Delta_l u^{N,\boldsymbol{n}}_{ik},\boldsymbol{m}^N(s),l)ds.
\end{aligned}\tag{10}$$

From Remark 1, we have $\sum_{jy}n^y_{ik}G^{N,ik}_{yj}(\alpha^{N,y}_{ik}(s))(u^{N,\boldsymbol{n}+\boldsymbol{e}_{jy},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))^2\leq\frac{K_2}{N}$. Then, since the terminal conditions are zero, we have

$$W^N_{ik}(t)\leq\frac{K_3}{N}+2\mathbb{E}\int_t^T(u^{N,\boldsymbol{n},l}_{ik}(s)-u^{N,\boldsymbol{n},l}_{ik}(s))(h(\Delta_l\boldsymbol{u}_{ik},\boldsymbol{m}(s),l)-h(\Delta_l u^{N,\boldsymbol{n}}_{ik},\boldsymbol{m}^N(s))ds,\tag{11}$$

where $K_3=K_2T$.

Using Proposition 1, h is Lipschitz function of $\Delta_l\boldsymbol{u}_{ik}$ and $\boldsymbol{m}_{ik}(t)$ $\forall(i,k)\in\mathcal{C}$. Hence,

$$(h(\Delta_l\boldsymbol{u}_{ik},\boldsymbol{m}(s),l)-h(\Delta_l u^{N,\boldsymbol{n}}_{ik},\boldsymbol{m}^N(s),l))\leq K_4(\sum_{(r,v)\in\mathcal{C}}||\frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}-\boldsymbol{m}_{\boldsymbol{rv}}(s)||+||u^{N,\boldsymbol{n}}_{ik}-u_{ik}||).\tag{12}$$

Then, from (11) and (12) and using the property $ab<a^2+b^2$, we have

$$\begin{aligned}
W^N_{ik}(t)&\leq\frac{K_3}{N}+K_4\mathbb{E}\int_t^T\sum_{(r,v)\in\mathcal{C}}||\frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}-\boldsymbol{m}_{\boldsymbol{rv}}(s)||^2+||u^{N,\boldsymbol{n}}_{ik}(s)-u^{ik}(s)||^2ds,\\
W^N_{ik}(t)&\leq\frac{K_3}{N}+K_4\mathbb{E}\int_t^T W^N_{ik}(s)+\sum_{(r,v)\in\mathcal{C}}V^N_{rv}(s)ds,\\
&\leq\frac{K_3}{N}+K_4\mathbb{E}\int_t^T W^N_{ik}(s)+\sum_{(r,v)\in\mathcal{C}}V^N_{rv}(s)ds,\\
&\leq\frac{C_1}{N}+C_1\mathbb{E}\int_t^T W^N_{ik}(s)+\sum_{(r,v)\in\mathcal{C}}V^N_{rv}(s)ds,
\end{aligned}\tag{13}$$

where $C_1=\max\{K_3,K_4\}$.

Appendix B: Proof of Lemma 3

By applying Dynkin's Formula (9) with $\varphi_l(j,\boldsymbol{n}_{ik},t)=(\boldsymbol{m}^l_{ik}(t)-\frac{\boldsymbol{n}^l_{ik}(t)}{N_{ik}})^2$ for all $l\in\mathcal{S}$, we get

$$V^N_{ik}(l,t)-\frac{(\nu^l_{ik})(1-\nu^l_{ik})}{2}=\mathbb{E}\int_0^t\frac{d\varphi_l}{dt}(l_{ik}(s),\boldsymbol{n}_{ik}(s),s)+A_{ik}\varphi^l(l_{ik}(s),\boldsymbol{n}_{ik}(s),s)ds,\tag{14}$$

where

$$\frac{d\varphi_l}{dt}(l_{ik}(s), \boldsymbol{n}_{ik}(s), s) = -2\Big(\frac{\boldsymbol{n}^l_{ik}(s)}{N_{ik}} - \boldsymbol{m}^l_{ik}(s)\Big) \sum_{j\in\mathcal{S}} G^{lj}_{ik}(\alpha^l_{ik}(s)) m^j_{ik}(s). \tag{15}$$

In what follows, we replace $\varphi^l(l_{ik}(s), \boldsymbol{n}_{ik}(s), s)$ by $\varphi^l(\boldsymbol{n}_{ik}(s), s)$ since φ_l is independent on $l_{ik}(s)$. Therefore,

$$\begin{aligned} A_{ik}\varphi(l_{ik}(s), \boldsymbol{n}_{ik}(s), s) &= \sum_{j\in\mathcal{S}} n^j_{ik} G^{ik}_{jl}(\alpha^{N,j}_{ik}(t))(\varphi_l(\boldsymbol{n}_{ik}(s) + \boldsymbol{e}_{lj}, h) - \varphi_l(\boldsymbol{n}_{ik}(s), s)) \\ &\quad + \sum_{j\neq l} n^l_{ik} G^{ik}_{lj}(\alpha^{N,l}_{ik}(t))(\varphi_l(\boldsymbol{n}_{ik}(s) + \boldsymbol{e}_{jl}, s) - \varphi_l(\boldsymbol{n}_{ik}(s), s)), \\ &= \Big(2(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)) + \frac{1}{N_{ik}}\Big) \sum_{j\neq l} \frac{n^j_{ik}(s)}{N_{ik}} G^{N,ik}_{jl}(\alpha^{N,j}_{ik}(s)) - (2(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)) - \frac{1}{N_{ik}}) \sum_{j\neq l} \frac{n^l_{ik}(s)}{N_{ik}} G^{N,ik}_{lj}(\alpha^{N,l}_{ik}(s)). \end{aligned} \tag{16}$$

Now, using the property that $\sum_{j\neq l} G^{N,ik}_{lj}(\alpha^{N,l}_{ik}(s)) = -G^{N,ik}_{ll}(\alpha^{N,l}_{ik}(s))$, we have

$$\begin{aligned} A_{ik}\varphi(l_{ik}(s), \boldsymbol{n}_{ik}(s), s) &= \Big(2(\frac{n^l_{ik}(s)}{N} - m^l_{ik}(s)) + \frac{1}{N}\Big) \sum_{j\neq l} \frac{n^j_{ik}(s)}{N} G^{N,ik}_{lj}(\alpha^{N,l}_{ik}(s)) \\ &\quad + (2(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)) - \frac{1}{N_{ik}}) \frac{n^l_{ik}(s)}{N} G^{N,ik}_{ll}(\alpha^{N,l}_{ik}(s)), \\ &\leq \Big(2(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)\Big) \sum_{j} \frac{n^j_{ik}(s)}{N} G^{N,ik}_{lj}(\alpha^{N,l}_{ik}(s)) + \frac{K_5}{N_{ik}} \end{aligned} \tag{17}$$

where the last equality follows since each transition rate is bounded. Thus,

$$\begin{aligned} &V^N_{ik}(l,t) \leq \mathbb{E}\int_0^t 2(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)) \sum_j \Big(\frac{n^j_{ik}(s)}{N} G^{ik}_{jl}(\alpha^j_{ik}(s)) - m^j_{ik}(s) G^{ik}_{jl}(\alpha^{N,j}_{ik}(s))\Big) + \frac{K_5}{N_{ik}}, \\ &= \mathbb{E}\int_0^t 2(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)) \sum_j \frac{n^j_{ik}(s)}{N_{ik}} (G^{ik}_{jl}(\alpha^{N,j}_{ik}(s)) - G^{ik}_{jl}(\alpha^j_{ik}(s)) \\ &+ G^{ik}_{jl}(\alpha^j_{ik}(s))((\frac{n^j_{ik}(s)}{N_{ik}} - m^j_{ik}(s)) ds + \frac{K_6}{N_{ik}}, \end{aligned} \tag{18}$$

where $K_6 = K_5 \cdot T$. Since in our game, the transitional rate is Lipchitz in $m_{ik}(t)$ $\forall$ (i,k) and in $\boldsymbol{u}_{ik}$ (according to Proposition 1), and using Remark 1 we have for $(i,k) = (i',k')$ ((i',k') is the class of the reference player)

$$\begin{aligned} &G^{ik}_{jl}(\alpha^{N,j}_{ik}(s)) - G^{ik}_{jl}(\alpha^j_{ik}(s)) \\ &\leq K_7(\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{\boldsymbol{rv}}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s) + \boldsymbol{e}_{jl}}{N_{rv}}||) + (||\boldsymbol{u}^{N,\boldsymbol{n}+e^{ik}_{jl}}_{ik}(s) - \boldsymbol{u}_{ik}||), \\ &\leq K_7(\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}||) + \frac{2}{N_{rv}} + (||\boldsymbol{u}^{N,\boldsymbol{n}+e^{ik}_{jl}}_{ik}(s) - \boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s)|| + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||), \\ &\leq K_7(\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}||) + \frac{2+2K_8}{N_{rv}} + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||. \end{aligned} \tag{19}$$

Also, for $(i,k) \neq (i',k')$, we have

$$
\begin{aligned}
& G^{ik}_{jl}(\alpha^{N,j}_{ik}(s)) - G^{ik}_{jl}(\alpha^{j}_{ik}(s)) \\
& \leq K_7 (\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{\boldsymbol{rv}}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s) - \boldsymbol{e}_l}{N_{rv}}||) + (||\boldsymbol{u}^{N,\boldsymbol{n}+e_{jl}}_{ik}(s) - \boldsymbol{u}_{ik}||), \\
& \leq K_7 (\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}}{N_{rv}}(s)||) + \frac{1}{N_{rv}} + (||\boldsymbol{u}^{N,\boldsymbol{n}-e^{ik}_l}_{ik}(s) - \boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s)|| + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||), \\
& \leq K_7 (\sum_{(r,v)\in\mathcal{C}} |||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}||) + \frac{2+2K_8}{N_{rv}} + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||. \qquad (20)
\end{aligned}
$$

By substituting (20) into (18), we get

$$
\begin{aligned}
V^N_{ik}(l,t) \leq\; & 2K_7 \mathbb{E} \int_0^t |\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)| \Big((\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}||) + \frac{K_9}{N_{rv}} + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)|| \Big) ds \\
& + \mathbb{E} \int_0^t 2 \Big(\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s) \Big) \sum_j G^{ik}_{jl}(\alpha_{ik}(t)) \Big(\frac{n^j_{ik}(s)}{N_{ik}} - m^j_{ik}(s) \Big) ds + \frac{K_6}{N}, \\
& \leq 2K_7 \mathbb{E} \int_0^t |\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)| \Big((\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}||) + \frac{K_{10}}{N_{rv}} + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)|| \Big) ds, \\
& + K_9 \mathbb{E} \int_0^t 2 \Big| \frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s) \Big| \sum_j \Big| \frac{n^j_{ik}(s)}{N_{ik}} - m^j_{ik}(s) \Big| ds,
\end{aligned}
\qquad (21)
$$

where $K_{10} = K_6 + 2T(1+K_7) + K_9$.

Let $(y,z) = \text{argmax}_{(r,v)} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{rv}(s)}{N_{rv}}||$. Thus,

$$
\begin{aligned}
|\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)| \Big((\sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{rv}(s) - \frac{\boldsymbol{n}_{\boldsymbol{rv}}(s)}{N_{rv}}||) & \leq \sum_{(r,v)\in\mathcal{C}} ||\boldsymbol{m}_{yz}(s) - \frac{\boldsymbol{n}_{yz}(s)}{N_{yz}}||^2, \\
& \leq |\mathcal{C}| ||\boldsymbol{m}_{yz}(s) - \frac{\boldsymbol{n}_{yz}(s)}{N_{rv}}||^2,
\end{aligned}
\qquad (22)
$$

and

$$
\begin{aligned}
|\frac{n^l_{ik}(s)}{N_{ik}} - m^l_{ik}(s)|.||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)|| & \leq ||\frac{\boldsymbol{n}_{ik}(s)}{N_{ik}} - \boldsymbol{m}_{ik}(s)||.||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||, \\
& \leq ||\frac{\boldsymbol{n}_{ik}(s)}{N_{ik}} - \boldsymbol{m}_{ik}(s)||^2 + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||^2. \qquad (23)
\end{aligned}
$$

From (21), (22), and (23), we have

$$
V^N_{ik}(l,t) \leq K_{11} \mathbb{E} \int_0^t ||\frac{\boldsymbol{n}_{ik}(s)}{N_{ik}} - \boldsymbol{m}_{ik}(s)||^2 + ||\boldsymbol{u}^{N,\boldsymbol{n}}_{ik}(s) - \boldsymbol{u}_{ik}(s)||^2 + ||\frac{\boldsymbol{n}_{yz}(s)}{N_{yz}} - \boldsymbol{m}_{yz}(s)||^2 + \frac{K_{10}}{N_{\max}}, \qquad (24)
$$

where $K_{11} = 2K_7 + K_9$ and $N_{\max} = \max_{(r,v)} N_{rv}$. Thus,

$$V_{ik}^N(t) \leq C_2 \mathbb{E} \int_0^t (V_{ik}^N(s) + W_{ik}^N(s) + V_{yz}^N(s)) ds + \frac{C_2}{N_{\max}}, \tag{25}$$

with $C_2 = \max\{K_{11}, K_{10}\}$.

□

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